13

MRC Technical Summary Report #1799

A FORCED QUASILINEAR WAVE EQUATION WITH DISSIPATION

J. A. Nohel

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

October 1977

(Received September 29, 1977)

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C. 20550

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

A FORCED QUASILINEAR WAVE EQUATION WITH DISSIPATION

J. A. Nohel

Technical Summary Report #1799
October 1977

ABSTRACT

We obtain sufficient conditions for the global existence, uniqueness, and continuous dependence of solutions of the quasilinear Cauchy problem (in one space dimension):

$$(*)\quad \begin{cases} y_{tt} + \alpha y_t - (\sigma(y_x))_x = g & (0 \le t < \infty,\ x \in \mathbb{R}) \\ y(0,x) = y_0(x),\ y_t(0,x) = y_1(x) & (x \in \mathbb{R})\ , \end{cases}$$

for smooth, small data y_0, y_1, and g. In (*) subscripts denote partial differentiation, $\alpha > 0$ is a constant, $\sigma : \mathbb{R} \to \mathbb{R}$, $g : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, $y_0, y_1 : \mathbb{R} \to \mathbb{R}$ are given sufficiently smooth functions, and $\sigma \in C^2(\mathbb{R})$ satisfies $\sigma(0) = 0$, $\sigma'(\xi) \ge \varepsilon > 0$ $(\xi \in \mathbb{R})$; the "genuinely nonlinear" problem $\sigma''(\xi) \not\equiv 0$ is of primary interest. The results can be used to study certain nonlinear Volterra functional differential equations arising in heat flow and viscoelastic motion for "materials with memory".

AMS (MOS) Subject Classifications: 35L60, 35L05, 35A05

Key Words: Quasilinear, Cauchy Problem, Wave Equation, Smooth Solutions, Global Existence, Uniqueness, Continuous Dependence, Characteristics, Riemann Invariants, Strictly Hyperbolic

Work Unit Number 1 (Applied Analysis)

Research sponsored by the United States Army under Grant No. DAAG29-77-G-0004 and Contract No. DAAG29-75-C-0024 and the National Science Foundation under Grant No. MSC75-21868.

SIGNIFICANCE AND EXPLANATION

This paper concerns the initial value problem for the one dimensional quasilinear wave equation with dissipation

$$(1)\qquad \begin{cases} y_{tt} + \alpha y_t - (\sigma(y_x))_x = g(t,x) & (0 < t < \infty, -\infty < x < \infty) \\ y(0,x) = y_0(x),\ y_t(0,x) = y_1(x) & (-\infty < x < \infty), \end{cases}$$

where the subscripts t, x denote partial differentiation, $\alpha > 0$ is a fixed constant, and the given real functions σ, g, y_0, y_1 are assumed to be sufficiently smooth. If $\sigma(\xi) = c^2\xi$, $-\infty < \xi < \infty$, c a constant, equation (1) is the linear wave equation with dissipation which can be solved explicitly by elementary methods.

However, there are problems arising in such applications as fluid dynamics, electrodynamics, elasticity, and others in which the linear problem $\sigma(\xi) = c^2\xi$ does not adequately model the physical situation. If, for example, $\sigma(\xi) = c^2\xi + f(\xi)$, where f is a smooth function with $f(0) = f'(0) = 0$ (e.g. $f(\xi) = \xi^2$), (1) can no longer be solved explicitly; yet it is important to obtain qualitative information about solutions. Before one can do this, one must know what types of solutions exist and for how long. If $\alpha = 0$, $g \equiv 0$ in (1), it is known that solutions can develop singularities in the first derivatives at a finite time t, even for arbitrarily smooth data $y_0(x)$, $y_1(x)$ (such solutions are called "shocks").

It is the purpose of this paper to obtain reasonable sufficient conditions for the global existence and uniqueness of smooth solutions of (1) for $\alpha > 0$. Our result states that such solutions exist for $0 \le t < \infty$, $-\infty < x < \infty$, provided the data functions y_0, y_1, g are sufficiently smooth and small (in a suitable norm); we also show that the solutions depend continuously on the data. Consequently, shock solutions do not arise in our situation. As an application we indicate briefly how our result can be used to discuss a problem arising in nonlinear heat flow and viscoelasticity.

The method of proof is technical and involves an extension of a method of T. Nishida who studied (1) with the forcing term $g \equiv 0$; Nishida did not consider the problem of the continuous dependence of the solution on the data y_0, y_1, and g.

A FORCED QUASILINEAR WAVE EQUATION WITH DISSIPATION

J. A. Nohel, Madison, Wisconsin*

1. Introduction. We study the global existence, uniqueness and continuous dependence on data of smooth solutions of the initial value problem

$$y_{tt} + \alpha y_t - (\sigma(y_x))_x = g \qquad (0 < t < \infty,\ x \in \mathbb{R}) \ , \tag{1.1}$$

$$y(0,x) = y_0(x),\ y_t(0,x) = y_1(x) \qquad (x \in \mathbb{R}) \ , \tag{1.2}$$

where the subscripts t, x denote partial differentiation, $\alpha > 0$ is a fixed constant, $\sigma : \mathbb{R} \to \mathbb{R}$, $g : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ and y_0, $y_1 : \mathbb{R} \to \mathbb{R}$ are given smooth functions. We shall assume throughout that

$$\sigma \in C^2(\mathbb{R}),\ \sigma(0) = 0,\ \sigma'(\xi) \geq \epsilon > 0 \qquad (\xi \in \mathbb{R};\ \epsilon > 0) \ ; \tag{σ}$$

the case $\sigma''(\xi) \not\equiv 0$ is of primary interest.

If $\alpha = 0$, $g \equiv 0$ it is known [4], [7] that solutions of the Cauchy problem (1.1), (1.2) will in general develop singularities in the first derivatives even for smooth data, and smooth solutions will not exist for large t. If $\alpha > 0$, $g \equiv 0$ Nishida [10], has established the existence and uniqueness of global smooth solutions of (1.1) for smooth and sufficiently small data (1.2) by a remarkably simple method.

It is the purpose of this note to (i) extend Nishida's method to obtain the global existence and uniqueness of smooth solutions of (1.1), (1.2) with $g \not\equiv 0$, and (ii) study the continuous dependence of solutions of (1.1), (1.2) on the data y_0, y_1, g. The result (i) is implicit in a recent paper of MacCamy [5]; however, his proof of the analogue of the important Lemma 2.3 below is not entirely complete. The result (ii) is new.

We remark that our results (i) and (ii) can be used to obtain a local existence and uniqueness result for smooth solutions of the functional differential equation

$$y_{tt} + \alpha y_t - (\sigma(y_x))_x = G(y) \qquad (0 \leq t \leq T,\ x \in \mathbb{R}) \ , \tag{1.3}$$

subject to the initial condition (1.2), for some $T > 0$. In (1.3) G is a given mapping defined on a suitable function space, and G satisfies a Lipschitz type condition. While limitations of space do not allow us to present this problem in detail, we point out that if $F(g)$ denotes the solution of (1.1), (1.2) on $[0,T] \times \mathbb{R}$, then a solution of (1.3), (1.2) is a fixed point of the composition map K defined by $K(y) = F(G(y))$. Such a fixed point can be found with the aid of our continuous dependence result for smooth solutions of (1.1), (1.2) for sufficiently small data in a manner similar to the method we used with Crandall

*Research sponsored by the United States Army under Grant No. DAAG29-77-G-0004 and Contract No. DAAG29-75-C-0024 and the National Science Foundation under Grant No. MSC75-21868.

in [1] to solve a functional differential equation in which, however, the underlying problem was an evolution equation of parabolic, rather than hyperbolic type. The details will be presented in a forthcoming joint paper with C. Dafermos.

The Cauchy problem (1.3), (1.2) has arisen in certain applications in heat flow and viscoelastic motion for "materials with memory" studied by MacCamy [5], [6]; the functional G has the form

$$(1.4) \qquad G(y)(t,x) = \Psi(t,x) + \beta y(t,x) + \int_0^t b(t-\tau)y(\tau,x)d\tau ,$$

where Ψ is a real smooth function on $[0,\infty) \times \mathbb{R}$ such that

$$\sup_{x\in\mathbb{R}} |\Psi(t,x)| \in L^1(0,\infty) \cap L^\infty(0,\infty), \ \sup_{x\in\mathbb{R}} |\Psi_x(t,x)| \in L^\infty(0,\infty) ,$$

$\beta > 0$ is a constant, and $b \in L^1(0,\infty;\mathbb{R})$, the value of G at (t,x) depends on the restriction of $y(\cdot,x)$ to $[0,t]$. In [5], [6] the interest is in the existence of global smooth solutions of the Cauchy problem (1.3), (1.4), (1.2); this is carried out by combining Nishida's method with certain delicate a priori estimates obtained by energy methods. However, the proof in [5], [6] appears to us to be incomplete, because the local existence problem for (1.3), (1.4), (1.2), which can be handled by the method outlined above, is essentially ignored.

In Section 2 we obtain the desired results for a "diagonal" strictly hyperbolic system of first order equations equivalent to (1.1), (1.2); the results for (1.1), (1.2) follow as an easy corollary and these are stated in Section 3. We acknowledge useful discussions with M. G. Crandall, C. Dafermos, and R. J. DiPerna during the preparation of this paper.

Finally, we mention related work of Matsumura [8], [9] received after the completion of this paper; the author generalizes Nishida's results for (1.1), (1.2) with $g \equiv 0$ from one space dimension to quasilinear hyperbolic equations in several space dimensions, and he obtains global existence of weak solutions and results concerning their decay (Nishida's method does not apply in this case).

2. Equivalent Systems and Preliminary Results. We assume that σ in (1.1) satisfies assumptions (σ). In addition, assume that g, and the initial functions y_0, y_1 in (1.1), (1.2) satisfy:

$$(g) \qquad g,\ g_x \in C([0,\infty) \times \mathbb{R}),\ g(t) = \sup_{x\in\mathbb{R}} |g(t,x)| \in L^\infty(0,\infty) \cap L^1(0,\infty) ,$$

$$g_1(t) = \sup_{x\in\mathbb{R}} |g_x(t,x)| \in L^\infty(0,\infty) ,$$

$$(I) \qquad y_0 \in \mathcal{B}^2(\mathbb{R}),\ y_1 \in \mathcal{B}^1(\mathbb{R}) ,$$

where $\mathcal{B}^m$ denotes the set of real functions with continuous and bounded derivatives up to and including order m.

Following Nishida [10] we reduce the Cauchy problem (1.1), (1.2) to the equivalent system (2.3) below. Putting $y_x = v$ and $y_t = w$ in (1.1), (1.2)

yields the equivalent Cauchy problem

$$(2.1)\qquad \begin{cases} v_t - w_x = 0,\ w_t - \sigma'(v)v_x + \alpha w = g & (0 < t < \infty,\ x \in \mathbb{R}) \\ v(0,x) = y_0'(x),\ w(0,x) = y_1(x) & (x \in \mathbb{R}) \end{cases} .$$

The eigenvalues of the matrix of (2.1)

$$\begin{pmatrix} 0 & 1 \\ -\sigma'(v) & 0 \end{pmatrix} .$$

are $\lambda = -\sqrt{\sigma'(v)}$, $\mu = \sqrt{\sigma'(v)}$; by assumptions (σ), λ and μ are real and distinct so that (2.1) is a strictly hyperbolic problem in the region $\{(v,w) : v \in \mathbb{R}, w \in \mathbb{R}\}$. To diagonalize (2.1) introduce the Riemann invariants

$$(2.2)\qquad r = w + \phi(v),\ s = w - \phi(v),\ \phi(v) = \int_0^v \sqrt{\sigma'(\xi)}\, d\xi \ ;$$

by (σ) the mapping $(v,w) \to (r,s)$ defined by (2.2) is one to one from $\mathbb{R} \times \mathbb{R}$ onto $\mathbb{R} \times \mathbb{R}$. A simple calculation shows that (2.1) is equivalent to the Cauchy problem for the diagonal, strictly hyperbolic system

$$(2.3)\qquad \begin{cases} r_t + \lambda r_x + \frac{\alpha}{2}(r+s) = g & (0 < t < \infty;\ x \in \mathbb{R}) \\ s_t + \mu s_x + \frac{\alpha}{2}(r+s) = g & \\ r(0,x) = r_0(x),\ s(0,x) = s_0(x) & (x \in \mathbb{R}) , \end{cases}$$

where by (2.2) $\lambda = \lambda(r-s)$, $\mu = \mu(r-s) \in C'(\mathbb{R})$, and where by (2.1)

$$(2.4)\qquad r_0(x) = y_1(x) + \phi(y_0'(x)),\ s_0(x) = y_1(x) - \phi(y_0'(x)) \qquad (x \in \mathbb{R}) \ ;$$

by assumptions (σ) and I, the initial data $r_0, s_0 \in \beta^1(\mathbb{R})$. It is also seen that if r, s is a smooth (β^1) solution of the problem (2.3) for $(t,x) \in \Omega \subseteq ([0,\infty) \times \mathbb{R})$, then y, determined by the relations $y_t = w(r,s)$, $y_x = v(r,s)$ (where v, w are uniquely determined by (2.2)), will be a smooth (β^2) solution of the Cauchy problem (1.1), (1.2) and conversely; we shall therefore deduce our results for (1.1), (1.2) from (2.3).

The following local result for (2.3) is known [2; Sec. 8], [3, Theorem VI]:

Lemma 2.1. _Let $r_0, s_0 \in \beta^1(\mathbb{R})$, let assumptions (σ) hold, and assume that $g, g_x \in \beta^0$ for $(t,x) \in [0,T] \times \mathbb{R}$, where $T > 0$. Then there exists a number $0 < T_1 \le T$ such that the Cauchy problem (2.3) has a unique smooth solution $r, s \in \beta^1([0,T_1] \times \mathbb{R})$._

The objective of the next two lemmas is to obtain apriori estimates on r, s, r_x, s_x (and hence by (2.3) on r_t, s_t), independent of T, which enable us to continue the local β^1-solution in t by a standard method.

Lemma 2.2. _Let the assumptions of Lemma 2.1 hold. In addition, assume that $g(t) = \sup_{x\in\mathbb{R}} |g(t,x)| \in L^1(0,\infty)$. Define the a priori constant $M_0 > 0$ by_

$$M_0 = r_0 + s_0 + 2\int_0^\infty g(\zeta)d\zeta,\ r_0 = \sup_{x\in\mathbb{R}} |r_0(x)|,\ s_0 = \sup_{x\in\mathbb{R}} |s_0(x)| .$$

For as long as the B^1-solution r, s of (2.3) exists one has

(2.5) $$\sup_{\substack{x\in\mathbb{R}\\ 0\le\tau\le t}} |r(\tau,x)| \le M_0,\quad \sup_{\substack{x\in\mathbb{R}\\ 0\le\tau\le t}} |s(\tau,x)| \le M_0 .$$

Sketch of Proof. The proof is similar to that of [10, Lemma 1], [5, Lemma 6.2]. Define the λ and μ characteristics of (2.3) respectively by

(2.6) $$x = x_1(t,\beta) = \beta + \int_0^t \lambda d\tau,\quad x = x_2(t,\gamma) = \gamma + \int_0^t \mu d\tau \quad (\beta,\gamma \in \mathbb{R}) ,$$

where $\lambda = \lambda[r(t,x_1(t,\beta)) - s(t,x_1(t,\beta))]$, $\mu = \mu[r(t,x_2(t,\gamma)) - s(t,x_2(t,\gamma))]$.

Let $' = \frac{\partial}{\partial t} + \lambda\frac{\partial}{\partial x}$, $` = \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial x}$ denote differentiation along the $\lambda$ and $\mu$ characteristics respectively, thus $r'(t,x_1) = \frac{d}{dt} r(t,x_1(t,\beta))$, $s`(t,x_2) = \frac{d}{dt} s(t,x_2(t,\gamma))$. Equations (2.3) become the ordinary differential equations

(2.7) $$\begin{cases} \frac{d}{dt} r(t,x_1(t,\beta)) + \frac{\alpha}{2} (r(t,x_1(t,\beta)) + s(t,x_1(t,\beta))) = g(t,x_1(t,\beta)) \\ \frac{d}{dt} s(t,x_2(t,\gamma)) + \frac{\alpha}{2} (r(t,x_2(t,\gamma)) + s(t,x_2(t,\gamma)) = g(t,x_2(t,\gamma)) \ ; \end{cases}$$

note that solutions of (2.7) will exist for as long as the slopes λ, μ of the characteristics $x_1(t,\beta)$ and $x_2(t,\gamma)$ remain bounded. Put

$$R(t) = \sup_{\substack{0\le\tau\le t\\ x\in\mathbb{R}}} e^{\frac{\alpha}{2}t} [|r(\tau,x)| + |s(\tau,x)|] ,$$

$$r_0 = \sup_{x\in\mathbb{R}} |r_0(x)|,\quad s_0 = \sup_{x\in\mathbb{R}} |s_0(x)| .$$

Integrate each of the equations (2.9) using $r(0,x_1(0,\beta)) = r_0(\beta)$, $s(0,x_2(0,\gamma)) = s_0(\gamma)$ (see (2.3), (2.6)), add the resulting equations and take absolute values; a standard argument yields the inequality

(2.8) $$R(t) \le r_0 + s_0 + \frac{\alpha}{2}\int_0^t R(\xi)d\xi + 2\int_0^t e^{\frac{\alpha}{2}\xi} g(\xi)d\xi .$$

Gronwall's inequality applied to (2.8) gives

(2.9) $$R(t) \le (r_0+s_0)e^{\frac{\alpha}{2}t} + 2e^{\frac{\alpha}{2}t} \int_0^t g(\xi)d\xi ,$$

and thus finally

(2.10) $$\sup_{\substack{0\le\tau\le t\\ x\in\mathbb{R}}} (|r(\tau,x)| + |s(\tau,x)|) \le M_0 ,$$

and the proof is complete.

Lemma 2.3. Let the assumptions of Lemma 2.1 and (g) be satisfied. Define the constant $D_1 > 0$ by

$$D_1 = r_0 + s_0 + \sup_{x\in\mathbb{R}} |r_0'(x)| + \sup_{x\in\mathbb{R}} |s_0'(x)| + \|g\|_{L^1(0,\infty)} + \|g\|_{L^\infty(0,\infty)} + \|g_1\|_{L^\infty(0,\infty)} .$$

For as long as the B^1*-solution* r, s *of* (2.3) *exists and if* D_1 *is sufficiently small, there exists a constant* $M_1 = M_1(D_1) > 0$ *where* $M_1(D_1) \to 0$ *as* $D_1 \to 0$, *such that*

$$\sup_{x\in\mathbb{R}} |r_x(t,x)| \le M_1, \quad \sup_{x\in\mathbb{R}} |s_x(t,x)| \le M_1 . \tag{2.11}$$

Sketch of Proof. (Compare [10, Lemma 2], [5, Lemma 6.3].) Differentiate the first equation in (2.3) obtaining (recall $\lambda = \lambda(r-s)$)

$$r_{xt} + \lambda r_{xx} = -\lambda_r r_x^2 - \lambda_s r_x s_x - \frac{\alpha}{2}(r_x+s_x) + g_x . \tag{2.12}$$

We remark that although Lemma 2.1 does not assert the existence of r_{xx} and r_{xt}, note that the left side of (2.12) is r_x' and this does exist for as long as the B^1-solution r, s of (2.3) exists. This observation also justifies the validity of equations (2.12)-(2.18) which follow. Since $\mu = -\lambda$ the second equation in (2.3) gives

$$s_x = \frac{s'}{2\lambda} + \frac{\alpha}{4\lambda}(r+s) - \frac{g}{2\lambda} \qquad \left(' = \frac{\partial}{\partial t} + \lambda\frac{\partial}{\partial x}\right) . \tag{2.13}$$

Define

$$h = \frac{1}{2}\log(-\lambda(r-s)) . \tag{2.14}$$

Differentiating h along the λ-characteristic and using $\lambda_s = -\lambda_r$ gives

$$h' = \frac{\lambda_r}{2\lambda}\left(-\frac{\alpha}{2}(r+s) + g - s'\right) . \tag{2.15}$$

Substitution of (2.13), (2.15) into (2.12) yields

$$r_x' + \left(\frac{\alpha}{2} + \lambda_r r_x + h'\right) r_x = -\frac{\alpha}{4\lambda}s' - \frac{\alpha^2}{8\lambda}(r+s) + \frac{\alpha}{4\lambda}g + g_x ,$$

or equivalently

$$(e^h r_x)' + \left(\frac{\alpha}{2} + \lambda_r r_x\right) e^h r_x = \left(-\frac{\alpha}{4\lambda}s' - \frac{\alpha^2}{8\lambda}(r+s) + \frac{\alpha}{4\lambda}g + g_x\right) e^h . \tag{2.16}$$

Define the function z by

$$z(r-s) = \int_0^{r-s} \frac{\alpha}{4\lambda(\xi)} e^{h(\xi)} d\xi ; \tag{2.17}$$

then $z' = -\dfrac{\alpha^2}{8\lambda} e^h(r+s) + \dfrac{\alpha}{4\lambda}e^h g - \dfrac{\alpha}{4\lambda}e^h s'$ and (2.16) becomes

$$(e^h r_x)' + \left(\frac{\alpha}{2} + \lambda_r r_x\right) e^h r_x = z' + e^h g_x . \tag{2.18}$$

To integrate (2.18) along the λ-characteristic put

$$\begin{cases} k(t) = \dfrac{\alpha}{2} + \lambda_r(r(t,x_1(t,\beta)) - s(t,x_1(t,\beta)))r_x(t,x_1(t,\beta)) \\ \rho(t) = r_x(t,x_1(t,\beta))\exp[h(t,x_1(t,\beta))] \\ p(t) = z'(t,x_1(t,\beta)) + g_x(t,x_1(t,\beta))\exp[h(t,x_1(t,\beta))] . \end{cases} \tag{2.19}$$

Then

$$(2.20) \qquad \rho(t) = \rho(0)\exp[-\int_0^t k(\tau)d\tau] + \int_0^t p(\xi)\exp[-\int_\xi^t k(\tau)d\tau]d\xi \cdot .$$

Suppose we can show that for any solution r, s of (2.3)

$$(2.21) \qquad |\lambda_r r_x| \le \frac{\alpha}{4} .$$

Then $k(t) = \frac{\alpha}{2} + \lambda_r(\cdot)r_x(\cdot) \ge \frac{\alpha}{4}$ and by an easy calculation

$$(2.22) \qquad |\rho(t)| \le |\rho(0)| + 3 \sup_{0\le\tau\le t} |z(\tau,x_1(\tau,\beta))|$$

$$+ \frac{4}{\alpha} \sup_{0\le\tau\le t} |g_x(\tau,x_1(t,\beta))\exp h(\tau,x_1(\tau,\beta)| .$$

We next show that (2.21) holds for any solution r, s of (2.3), provided the constant $D_1 > 0$ is sufficiently small. Indeed, at $t = 0$ and for any $\beta \in \mathbb{R}$ $\lambda_r(r(0,\beta) - s(0,\beta))r_x(0,\beta)$ satisfies

$$(2.23) \qquad |r_0'(\beta)\lambda_r(r_0(\beta) - s_0(\beta))| \le \sup_{x\in\mathbb{R}} |r_0'(x)\lambda_r(r_0(x) - s_0(x))| < \frac{\alpha}{4}$$

provided $D_1 > 0$ is sufficiently small. By (2.19) $\lambda_r r_x = \lambda_r e^{-h}\rho$ and by (σ), (2.14) and Lemma 2.2 $|\lambda_r e^{-h}|$ is uniformly bounded in (t,x) by a constant $K_1(D_1) > 0$, where $K_1(D_1) \to 0$ as $D_1 \to 0$, for any solution r, s of (2.3). By (2.14), (2.17), (2.22) and Lemma 2.2 the quantity ρ is uniformly bounded for any solution r, s of (2.3) as follows:

$$(2.24) \qquad |\rho(t)| \le |\rho(0)| + 3 \sup_{\substack{0\le\tau\le t\\ x\in\mathbb{R}}} |z[r(\tau,x) - s(\tau,x)]|$$

$$+ \frac{4}{\alpha} \sup_{\substack{0\le\tau\le t\\ x\in\mathbb{R}}} \sqrt{-\lambda[r(\tau,x) - s(\tau,x)]}\, |g_x(\tau,x)|$$

$$\le K_2(D_1), \quad \text{where } K_2(D_1) \to 0 \text{ as } D_1 \to 0 .$$

Therefore $|\lambda_r r_x| \le K_1(D_1)K_2(D_1)$ uniformly in (t,x) for any solution r, s of (2.3). The assertion (2.21) holds at $t = 0$, for D_1 sufficiently small (by (2.23)). Choosing D_1 smaller if necessary so that $K_1(D_1)K_2(D_1) \le \frac{\alpha}{4}$, we conclude that (2.21) continues to hold for as long as the solution r, s exists. Returning to (2.22), (2.24) and using $r_x = \rho e^{-h}$ establishes the estimate for r_x in (2.11); a similar argument yields the estimate for s_x and completes the proof of Lemma 2.3.

The a priori estimates of Lemmas 2.2 and 2.3, together with $\|g\|_{L^\infty(0,\infty)} < \infty$ yield uniform a priori estimates for r_t, s_t, for any solution r, s of (2.3), provided $D_1 > 0$ is sufficiently small. Then Lemmas 2.1, 2.2, 2.3 and a standard continuation argument give the first part of the following global result for the Cauchy problem (2.3).

Theorem 2.1. Let the assumption (σ), (g) be satisfied, and let the initial data $r_0, s_0 \in \beta^1(\mathbb{R})$. If the constant D_1 (see Lemma 2.3) is sufficiently small, then the Cauchy problem (2.3) has a unique β^1-solution r, s for $0 \leq t < \infty$, $x \in \mathbb{R}$ and the a priori estimates (2.5), (2.11) are satisfied for $0 \leq t < \infty$.

Let the above assumptions be satisfied by initial data r_0, s_0 and $\bar{r}_0, \bar{s}_0$ and forcing functions $g, \bar{g}$; denote by r, s and $\bar{r}, \bar{s}$ the corresponding β^1-solutions of (2.3) on $[0,\infty) \times \mathbb{R}$. Define

$$\zeta(t) = \sup_{x\in \mathbb{R}} |r(t,x) - \bar{r}(t,x)| + \sup_{x\in \mathbb{R}} |s(t,x) - \bar{s}(t,x)| .$$

Then there exists a constant $M_2 = M_2(\sigma, M_0) > 0$ such that

$$(2.25) \quad \zeta(t) \leq e^{2M_2M_1t}\left(\zeta(0) + \int_0^t e^{-2M_2M_1\tau} \sup_{x\in \mathbb{R}} |g(\tau,x) - \bar{g}(\tau,x)| d\tau\right) \quad (0 \leq t < \infty) ,$$

where M_0, M_1 are the bounds in (2.5), (2.11) respectively.

Remark. The continuous dependence result (2.25) also holds for local solutions r, s and $\bar{r}, \bar{s}$ on $[0,T_1] \times [0, \ )$ of the Cauchy problem (2.3) (see Lemma 2.1), but only for $0 \leq t \leq T_1$.

Proof of Theorem 2.1. It remains only to prove (2.25). If r, s and $\bar{r}, \bar{s}$ are β^1-solutions of (2.3) for the situation in the theorem, one has

$$(2.26) \quad \begin{cases} (r-\bar{r})_t + \lambda r_x - \bar{\lambda}\bar{r}_x = -\frac{\alpha}{2}[(r-\bar{r}) + (s-\bar{s})] + g - \bar{g} \\ (s-\bar{s})_t + \mu s_x - \bar{\mu}\bar{s}_x = -\frac{\alpha}{2}[(r-\bar{r}) + (s-\bar{s})] + g - \bar{g} \end{cases} \quad \begin{matrix} 0 < t < \infty \\ x \in \mathbb{R} \end{matrix}$$

subject to the initial conditions

$$(2.27) \quad r(0,x) - \bar{r}(0,x) = r_0(x) - \bar{r}_0(x), \ s(0,x) - \bar{s}(0,x) = s_0(x) - \bar{s}_0(x) \quad (x \in \mathbb{R})$$

where $\lambda = \lambda(r-s)$, $\mu = \mu(r-s)$, $\bar{\lambda} = \lambda(\bar{r}-\bar{s})$, $\bar{\mu} = \mu(\bar{r}-\bar{s})$. But

$$\lambda r_x - \bar{\lambda}\bar{r}_x = \lambda(r-\bar{r})_x + (\lambda-\bar{\lambda})\bar{r}_x$$
$$\mu s_x - \bar{\mu}\bar{s}_x = \mu(s-\bar{s})_x + (\mu-\bar{\mu})\bar{s}_x ;$$

Therefore (2.26) can be written as

$$(2.28) \quad \begin{cases} (r-\bar{r})_t + \lambda(r-\bar{r})_x = -\frac{\alpha}{2}[(r-\bar{r}) + (s-\bar{s})] + g - \bar{g} - (\lambda-\bar{\lambda})\bar{r}_x \\ (s-\bar{s})_t + \mu(s-\bar{s})_x = -\frac{\alpha}{2}[(r-\bar{r}) + (s-\bar{s})] + g - \bar{g} - (\mu-\bar{\mu})\bar{s}_x . \end{cases}$$

Recalling the definitions of λ, μ and using (σ) and the mean value theorem one has

$$\lambda - \bar{\lambda} = \lambda(r-s) - \lambda(\bar{r}-\bar{s}) = \frac{d\lambda}{d\xi}(\bar{r} - \bar{s} + \theta_1[r - s - (\bar{r}-\bar{s})])(r - s - (\bar{r}-\bar{s}))$$
$$\mu - \bar{\mu} = \mu(r-s) - \mu(\bar{r}-\bar{s}) = \frac{d\mu}{d\xi}(\bar{r} - \bar{s} + \theta_2[r - s - (\bar{r}-\bar{s})])(r - s - (\bar{r}-\bar{s})) ,$$

for some $0 < \theta_1, \theta_2 < 1$. Therefore, (2.28) becomes

$$(2.29)\quad \begin{cases} (r-\bar r)' + \frac{\alpha}{2}(r-\bar r) = -\frac{\alpha}{2}(s-\bar s) - \frac{d\lambda}{d\zeta}(\cdot)(r-\bar r-(s-\bar s))\bar r_x + g - \bar g \\ (s-\bar s)^{`} + \frac{\alpha}{2}(s-\bar s) = -\frac{\alpha}{2}(r-\bar r) - \frac{d\mu}{d\zeta}(\cdot)(r-\bar r-(s-\bar s))\bar s_x + g - \bar g , \end{cases}$$

where $' = \frac{\partial}{\partial t} + \lambda\frac{\partial}{\partial x}$, $` = \frac{\partial}{\partial t} + \mu\frac{\partial}{\partial x}$. We next note that assumption (σ) and Lemmas 2.2 and 2.3 imply the existence of a constant $M_2 = M_2(\sigma, M_0)$ such that

$$(2.30)\qquad \left|\frac{d\lambda}{d\zeta}(\cdot)\bar r_x\right| \le M_2M_1, \quad \left|\frac{d\mu}{d\zeta}(\cdot)\bar s_x\right| \le M_2M_1 ,$$

uniformly in $(t,x) \in [0,\infty) \times \mathbb{R}$ where M_1 is the bound in Lemma 2.3. Integrating the first equation in (2.29) along any λ-characteristic and the second along any μ-characteristic and making simple estimates one obtains the pair of inequalities

$$e^{\frac{\alpha}{2}t}\sup_{x\in\mathbb{R}}|r(t,x)-\bar r(t,x)| \le \sup_{x\in\mathbb{R}}|r_0(x)-\bar r_0(x)| + \int_0^t e^{\frac{\alpha}{2}\tau}\sup_{x\in\mathbb{R}}|g(\tau,x)-\bar g(\tau,x)|d\tau$$

$$+ \int_0^t e^{\frac{\alpha}{2}\tau}\left(\frac{\alpha}{2}+M_2M_1\right)\sup_{x\in\mathbb{R}}|s(\tau,x)-\bar s(\tau,x)|d\tau + M_1M_2\int_0^t e^{\frac{\alpha}{2}\tau}\sup_{x\in\mathbb{R}}|r(\tau,x)-\bar r(\tau,x)|d\tau ,$$

$$e^{\frac{\alpha}{2}t}\sup_{x\in\mathbb{R}}|s(t,x)-\bar s(t,x)| \le \sup_{x\in\mathbb{R}}|s_0(x)-\bar s_0(x)| + \int_0^t e^{\frac{\alpha}{2}\tau}\sup_{x\in\mathbb{R}}|g(\tau,x)-\bar g(\tau,x)|d\tau$$

$$+ \int_0^t e^{\frac{\alpha}{2}\tau}\left(\frac{\alpha}{2}+M_2M_1\right)\sup_{x\in\mathbb{R}}|r(\tau,x)-\bar r(\tau,x)|d\tau + M_2M_1\int_0^t e^{\frac{\alpha}{2}\tau}\sup_{x\in\mathbb{R}}|s(\tau,x)-\bar s(\tau,x)|d\tau .$$

Adding these inequalities one obtains on using the definition of ζ

$$(2.31)\qquad \zeta(t)e^{\frac{\alpha}{2}t} \le \zeta(0) + 2\int_0^t e^{\frac{\alpha}{2}\tau}\sup_{x\in\mathbb{R}}|g(\tau,x) - \bar g(\tau,x)|d\tau$$

$$+ \int_0^t \left(\frac{\alpha}{2} + 2M_1M_2\right)e^{\frac{\alpha}{2}\tau}\zeta(\tau)d\tau \qquad (0 \le t < \infty) .$$

Finally, applying Gronwall's inequality to (2.31) yields the result (2.25) completing the proof.

3. Global Existence, Uniqueness, and Continuous Dependence for the Cauchy Problem (1.1), (1.2). As an immediate consequence of Theorem 2.1 and of the equivalence of the Cauchy problems (1.1), (1.2) and (2.3), (2.4) we obtain the main result of this paper.

Theorem 3.1. *Let the assumptions (σ), (g), (I) be satisfied. Define the constant*

$$(3.1)\quad D = \sup_{x\in\mathbb{R}}|y_0'(x)| + \sup_{x\in\mathbb{R}}|y_1(x)| + \sup_{x\in\mathbb{R}}|y_0''(x)| + \|g\|_{L^1(0,\infty)} + \|g\|_{L^\infty(0,\infty)} + \|g_1\|_{L^\infty(0,\infty)} .$$

If D is sufficiently small, then the Cauchy problem (1.1), (1.2) has a unique $\mathcal{B}^2$-solution y on $[0,\infty) \times \mathbb{R}$, and the solution y satisfies the a priori estimates:

(3.2) $$\sup_{x\in\mathbb{R}} |y_x(t,x)|,\ \sup_{x\in\mathbb{R}} |y_t(t,x| \le \sup_{x\in\mathbb{R}} |y_0'(x)| + \sup_{x\in\mathbb{R}} |y_1(x)| + 2\int_0^\infty g(\xi)d\xi = M_0 \quad (0 \le t < \infty),$$

and there exists a constant $M_1 = M_1(D) > 0$ (which $\to 0$ as $D \to 0$) such that

(3.3) $$\sup_{x\in\mathbb{R}} |y_{xx}(t,x)|,\ \sup_{x\in\mathbb{R}} |y_{xt}(t,x)|,\ \sup_{x\in\mathbb{R}} |y_{tt}(t,x)| \le M_1 \quad (0 \le t < \infty).$$

Let, in addition, the assumptions (g) and (I) be satisfied also by functions $\bar{g}$ and $\bar{y}_0$, $\bar{y}_1$ and let $\bar{y}$ denote the corresponding β^1-solution on $[0,\infty) \times \mathbb{R}$. Define

$$\zeta(t) = \sup_{x\in\mathbb{R}} |y_x(t,x) - \bar{y}_x(t,x)| + \sup_{x\in\mathbb{R}} |y_t(t,x) - \bar{y}_t(t,x)|;$$

then there exists a constant $M_2 = (\sigma, M_0) > 0$ such that

(3.4) $$\zeta(t) \le e^{2M_2M_1t}\left(\zeta(0) + \int_0^t e^{-2M_2M_1\tau} \sup_{x\in\mathbb{R}} |g(\tau,x) - \bar{g}(\tau,x)|d\tau\right) \quad (0 \le t < \infty),$$

where M_0, M_1 are bounds in (3.2), (3.3) respectively; moreover (using $y(t,x) = y_0(x) + \int_0^t y_t(\tau,x)d\tau$)

$$\sup_{x\in\mathbb{R}} |y(t,x) - \bar{y}(t,x)| \le \sup_{x\in\mathbb{R}} |y_0(x) - \bar{y}_0(x)|$$
$$+ \frac{e^{2M_1M_2t}}{M_1M_2}\left(\sup_{x\in\mathbb{R}} |y_0'(x) - \bar{y}_0'(x)| + \sup_{x\in\mathbb{R}} |y_1(x) - \bar{y}_1(x)|\right.$$
$$\left. + \int_0^t e^{-2M_1M_2\tau} \sup_{x\in\mathbb{R}} |g(\tau,x) - \bar{g}(\tau,x)|d\tau\right) \quad (0 \le t < \infty).$$

References

1. M. G. Crandall and J. A. Nohel, An abstract functional differential equation and a related nonlinear Volterra equation. Israel J. Math. (to appear; also Math. Res. Center, Univ. of Wisconsin, Madison, Tech. Summary Report No. 1765, July 1977).

2. A. Douglis, Some existence theorems for hyperbolic systems of partial differential equations in two independent variables. Comm. Pure Appl. Math. 5 (1952), 119-154.

3. P. Hartmann and A. Wintner, On hyperbolic partial differential equations. American J. Math. 74 (1952), 834-864.

4. P. D. Lax, Development of singularities of solutions of nonlinear hyperbolic partial differential equations. J. Math. Phys. 5 (1964), 611-613.

5. R. C. MacCamy, An integro-differential equation with applications in heat flow. Q. Appl. Math. 35 (1977), 1-19.

6. R. C. MacCamy, A model for one-dimensional, nonlinear viscoelasticity. Q. Appl. Math. 35 (1977), 21-33.

7. R. C. MacCamy and V. Mizel, Existence and nonexistence in the large of solutions of quasilinear wave equations. Arch. Rational Mech. and Anal. 25 (1967), 299-320.

8. A. Matsumura, Global existence and asymptotics of the solutions of the second-order quasilinear hyperbolic equations with first order dissipation, (to appear).

9. A. Matsumura, Energy decay of solutions of dissipative wave equations (to appear).

10. T. Nishida, Global smooth solutions for the second-order quasilinear wave equations with the first-order dissipation (unpublished).

Author's address: Mathematics Research Center, University of Wisconsin-Madison, 610 Walnut Street, Madison, Wisconsin 53706 U.S.A.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 1799 — MRC-TSR-1799 (14)

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle): (6) A FORCED QUASILINEAR WAVE EQUATION WITH DISSIPATION.

5. TYPE OF REPORT & PERIOD COVERED: (9) Technical Summary Report, no specific reporting period

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s): (10) J. A. Nohel

8. CONTRACT OR GRANT NUMBER(s): (15) DAAG29-77-G-0004, DAAG29-75-C-0024, ~~MCS75-21868~~

9. PERFORMING ORGANIZATION NAME AND ADDRESS: Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 1 (Applied Analysis)

11. CONTROLLING OFFICE NAME AND ADDRESS: See Item 18 below.

12. REPORT DATE: (11) October 1977

13. NUMBER OF PAGES: 10 (12) 15P.

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report): UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Quasilinear	Uniqueness
Cauchy Problem	Continuous Dependence
Wave Equation	Characteristics
Smooth Solutions	Riemann Invariants
Global Existence	Strictly Hyperbolic

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We obtain sufficient conditions for the global existence, uniqueness, and continuous dependence of solutions of the quasilinear Cauchy problem (in one space dimension):

$$(*)\quad \begin{cases} y_{tt} + \alpha y_t - (\sigma(y_x))_x = g & (0 \le t < \infty,\ x \in \mathbb{R}) \\ y(0,x) = y_0(x),\ y_t(0,x) = y_1(x) & (x \in \mathbb{R}), \end{cases}$$

for smooth, small data y_0, y_1, and g. In (*) subscripts denote partial

DD FORM 1473, 1 JAN 73 — EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

221 200

JOB

20. ABSTRACT - Cont'd.

differentiation, $\alpha > 0$ is a constant, $\sigma : \mathbb{R} \to \mathbb{R}$, $g : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, $y_0, y_1 : \mathbb{R} \to \mathbb{R}$ are given sufficiently smooth functions, and $\sigma \in C^2(\mathbb{R})$ satisfies $\sigma(0) = 0$, $\sigma'(\xi) \geq \varepsilon > 0$ $(\xi \in \mathbb{R})$; the "genuinely nonlinear" problem $\sigma''(\xi) \not\equiv 0$ is of primary interest. The results can be used to study certain nonlinear Volterra functional differential equations arising in heat flow and viscoelastic motion for "materials with memory".